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Algebraic Thinking

in Adult Education

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Introduction

A world where employers "will be looking for the most competent, creative, and most innovative" workers is predicted in *Tough Choices or Tough Times*, a report by the New Commission on the Skills of the American Workforce (2006). The report notes as well that literacy in which "mathematical reasoning will be no less important than facts" is critical not only for top professionals, but also for those at all skill levels in the workforce. As more industries adopt new technologies, a solid understanding of mathematics and science will be needed to realize the full potential of these technologies in new situations.

Industries such as biotechnology, geospatial, health care, financial services and the skilled trades "can't find enough workers with the right skills for these high-skilled, good-paying jobs" (U.S. Department of Labor 2005). These are examples of industries that need employees with mathematical knowledge and skills. The greatest rewards will go to those who are comfortable with ideas and abstractions, can adapt flexibly to changes, and can generalize and synthesize.

What kind of mathematics instruction for adults would enable them to meet the demands of this predicted future? Conceptual understanding and problem-solving ability are critical proficiencies that should be the cornerstones of a revised set of basic skills. Further, this paper proposes that *algebraic reasoning*, a way of thinking that reflects the core skills and underlying principles supporting number relationships and operations, be integrated early into all levels of arithmetic instruction. Although there are various conceptions of algebraic thinking in the field, in this paper we use the term to mean thinking that involves

- looking for structure (patterns and regularities) to make sense of situations
- generalizing beyond the specific by using symbols for variable quantities
- representing relationships systematically with tables, graphs, and equations
- reasoning logically to address/solve new problems

Providing opportunities to generalize the relationships and properties of arithmetic at an early stage of learning increases students' chance for success in a formal algebra class. Further, algebraic reasoning gives a logical coherence to arithmetic procedures, which are often perceived to be arbitrary and incoherent.

We also propose that formal algebra instruction for adults emphasize *modeling* (analyzing and representing real situations with mathematical structures), which is fundamental to applying mathematical concepts to everyday life and workplace situations. We believe that this approach is tailor-made for adult students.

Both the K–12 and adult education systems already have begun to take steps to integrate algebraic reasoning at all levels. For K–12 instruction, the National Council of Teachers of Mathematics (NCTM) published a list (NCTM 2006) of focal points for mastery at each grade level from kindergarten to eighth grade, including an early algebra strand emphasizing recognition of the properties of the numbers and operations students are learning. At the adult level, there is a new focus on preparing for post-secondary education and the training required for jobs in high-growth, high-demand industries. In their new adult mathematics standards, many states recommend introducing algebraic thinking and data analysis to learners at all levels. And some go beyond the traditional goal of passing the GED (General Educational Development) test by offering programs that assist students in their transition to community college or vocational training.

Why is Algebra Important in Adult Education?

Mathematics proficiency is needed to satisfy formal academic requirements for advancement as well as to meet the genuine skill demands of home and work. "It's the math that's killing us," notes Dr. Donna McKusik, director of developmental education at the Community College of Baltimore County, referring to the many entering students with test scores indicating that they need to pass a high-school-level algebra course to proceed toward their career goals. She observes that the course is a real stumbling block for many; they lose confidence, dropping out not just of math class but also of college, even though their skills in other subjects are up to par (Schemo, *New York Times*, Sept. 2, 2006).

Even among those who have finished college and are starting their careers, there is a sobering lack of practical mathematical ability. According to the *National Survey of America's College Students* (American Institutes for Research 2006), which assessed prose, document, and quantitative literacy, graduating college students struggled most with quantitative literacy. In fact, 30 percent of students earning two-year degrees and 20 percent of those earning four-year degrees have only basic quantitative literacy skills. These students were unable to complete such tasks as calculating the total cost of office supplies or estimating whether their car has enough gas left to make it to the next gas station.

Solid mathematical skills and understanding are essential in virtually all aspects of life, and they are formal requirements in many of the roles adults play. The following examples illustrate the use of —and need for— algebra in everyday situations, formal education, and the workplace.

Personal Financial Literacy and Decision Making

Full participation in today's society involves a genuine grasp of the mathematical concepts that studying algebra provides. Quantitative demands can be both complex and pervasive, reaching into an individual's daily life as citizen, worker, parent, and consumer. Often it is the big ideas of algebra, not the procedural details, which people draw upon when making wise decisions.

For example, the concept that change often occurs over time according to predictable patterns based on mathematical relationships is a valuable tool for decision making. A young person who understands the power of compounding (i.e., the shape of the graph of exponential growth) when saving and investing may be more likely to restrict spending and begin saving at an early age to build lifetime assets. More important, those who recognize the painful side of compounding can avoid building up credit card debt, as interest charges are compounded and debt quickly grows to high levels. To be clear, although the skills and procedures of algebra are important, it is often the more sophisticated understanding of mathematical relationships imparted by algebra that is useful in navigating life's decision-making challenges.

Academic Requirements

Completing an algebra course is often a credentialing requirement, a hurdle to jump on the way to further study. Adult students first meet the formal requirement for algebra in their preparation for the GED test, the goal of many who come to adult education classes. Since one or often two courses in algebra are included in requirements for high school graduation in many states, algebra also is included on the GED mathematics test. The next algebra hurdle appears when students take college mathematics placement tests. At least one year of algebra is required for students intending to pursue most two-year technical certificates or degrees, as well as for those planning to transfer to a four-year college.

Students not sufficiently proficient in mathematics to proceed to college-level course work are required to enroll in courses at the high school, or even middle school, level offered as developmental courses. At most colleges, a developmental algebra class "covers" a full year of high school algebra in one semester and typically does not carry college credit. As a result, 57 percent of two-year college students take (and often repeat) mathematics courses without gaining any college credits (Lutzer et al. 2007). These students pay college tuition for courses offered free in high school (and free or for a nominal fee in adult education programs). Even more distressing is the fact that many of these students deplete their financial aid in developmental courses. One study found that only one-quarter of those referred to a sequence of developmental classes had successfully completed them by the end of their third year (Clery 2006).

Entrance-to-Employment Requirements

The formal entrance requirements for careers in many high-growth industries include at least an associate's degree. The 50 fastest-growing occupations from 2006-2016 (http://www.careerinfonet.org) include only 12 occupations that do *not* require postsecondary certificates or degrees. (Preparation for these occupations involves various levels of on-the-job training.) All but one of these 12 non-degree occupations (gaming managers) were in the lower two quartiles of earnings, which translates to an annual salary of less than $28,570, less than the income necessary to support a family in most parts of the country.

This trend can also be seen by examining the 50 occupations with *declining* employment from 2006–2016 (http://www.careerinfonet.org). Most of these occupations (47 out of 50) require "on-the-job training." Clearly, satisfactory employment opportunities are likely to be even more limited than in previous years for those who have not demonstrated competence in algebra, a requirement for a post secondary certificate or degree.

On-the-Job Requirements

The benefits of studying algebra are sometimes subtle. In the workplace, workers often are not aware when they use an overarching mathematical concept. For example, researchers have shown that the concept of proportionality was applied in nearly every workplace they studied (Hoyles et al. 2002; Marr and Hagston 2007; Selden and Selden 2001). Nurses use proportionality when determining the correct dose of medication, and cosmetologists use it when mixing solutions, but few recognize that they are using "school math" because the mathematical ideas are so deeply embedded into the context of the job. Adults often say they have never used the algebra they learned in school. That may be true for the rote aspects of manipulating symbols, but they likely are using the mathematical reasoning and problem-solving aspects of algebra unconsciously.

Technological advances affect the mathematics required in the workplace in different ways. On the one hand, computers have lessened the requirements for workers themselves to be precise in routine work, be it calculations in clerical positions or measurement on an assembly line. Digital devices also perform monitoring tasks, assessing and graphing the state or quality of a process. On the other hand, mathematical requirements have increased because workers need to be comfortable using these devices and monitoring the outcomes.

Selden and Selden (2001) describe how the proliferation of graphical data has created specific demands involving the algebraic topic of graphing with rectangular coordinates. Management information, formerly the domain of higher-level workers, must now be interpreted by those who work "on the line." This is true in many jobs, from home health aides to factory production workers. Workers need to understand the relationship between the conventional aspects of Cartesian graphing (e.g., points, slopes, intercepts, intersections) and how to reason within that abstract system, so they can recognize what the graphs represent in the real context. For example, a steep line indicates a rapid change in whatever is being tracked, and the context determines if this change is normal or abnormal.

Office workers and production workers alike need to recognize out-of-whack results before they are disseminated. Using spreadsheets to make calculations with large amounts of data eliminates the need to do the task by hand, but also increases the impact of one mistake. While the need for precise paper-and-pencil calculation on the job may have decreased with the onset of automated systems, the ability to estimate reasonable answers has become more valuable in checking for errors that could have widespread effects.

The ability to write a mathematical expression or algebraic equation so that a software program will handle data as one intends is also critical. Creating and adjusting an abstract model requires deeper understanding of mathematical relationships, not just the ability to deal with the details of a specific example. Algebraic reasoning with symbols is needed to determine the equivalence of various expressions that could be used as the formula for a certain operation in a spreadsheet. For example, a 5 percent increase in a quantity (x) could be entered as $x + .05x$ or as $1.05x$.

The ability to remain flexible in the methods used for computation allows workers to adapt to new requirements. Researchers such as Noss, Hoyles and Pozzi (2000) and Scribner (1984) have noted that workers used "idiosyncratic" methods for computing in the workplace and passed them on to new workers. Their invented methods were effective in specific situations and easier for them than the formal procedures learned in school. Although they may not have understood the principles that made their invented methods work, they could recognize consistent patterns and devise a method for their situation. When changes occur in the workplace, however, workers must have sufficient understanding to be able to re-evaluate their method to see if it is still appropriate and devise a different method if necessary. Flexible thinking, not mindless adherence to a procedure, is required in such situations.

Recent Scholarship about Algebra

The research described here is based on work with children. Currently, there is no comparable body of research for adults. Based on our years of teaching mathematics to adults in various settings and our experience in assessment and research, we are extrapolating from the existing research, which we believe provides insight into algebra teaching and learning for adults. Schmitt (2003) has observed that adults who are returning to mathematics study "possess some, but never all, of the characteristics of the more frequently studied groups" (p. 67).

Various researchers have offered different descriptions of algebra. For example, Usiskin (1995) stated that algebra uses symbolic language to describe real and hypothetical patterns and includes generalized arithmetic, a means to solve problems, the study of relationships, and the study of mathematical structures. Romberg and Spence (1995) stressed that "algebra is a tool for making sense of the world — for making predictions and for making inferences about things that you cannot measure or count" (p. 186).

James Kaput (2007) has offered a conceptualization that includes three strands of algebra within which the processes of generalizing and conventional symbol manipulation occur:

1. Generalizing arithmetic and quantitative reasoning, with particular emphasis on symbols, expressions, and equations;

2. Studying functions, relations, and joint variation, with the use of a wide range of representations, including equations, tables, graphs and "various pedagogical systems such as 'function machines'" (p. 14); and

3. Using modeling to generalize and express patterns or regularities in situations from within or outside of mathematics or to move from specific examples to more general forms that highlight relationships.

The predominant content and instructional approaches current in school algebra courses in the United States prioritize different aspects of Kaput's components. Kieran (2007) describes "traditional" programs as having a strong symbolic orientation and focus on developing formal procedures to simplify and manipulate expressions, equations, inequalities, systems of equations, and polynomials. The emphasis is on recognizing forms and mastering transformational processes. Reform-oriented programs emphasize the study of functions, using letters to represent variables and equations to describe real-world situations or activities. Algebraic notation and various representations are used to describe how quantities vary with each other and depend on each other. Algebraic procedures are used to operate within the mathematical system, but they do not define it.

Definitive research on how algebra is best learned and taught is not available, and research on different approaches has been varied and sometimes contradictory (National Mathematics Advisory Panel 2008). In a year-long qualitative research study, however, Chazan (2000) found that instruction emphasizing functions was particularly engaging and meaningful for at-risk high school students. Similarly, the modeling approach that emerges from experiential learning has been used by the Algebra Project and found to be accessible and effective for African-American children who might otherwise be marginalized (Davis et al. 2007; Moses and Cobb 2001). These findings suggest that it might be fruitful to employ a modeling/functions approach with adult learners, as they resemble the populations in these studies.

Problematic Algebraic Concepts

Numerous research studies have examined common aspects of each of the strands that have consistently caused students difficulty, with some consensus emerging on the mechanisms that contribute to the difficulties. Below we summarize some of the findings related to three algebraic "big ideas," or concepts, that may be especially consequential for enhancing adult algebra learning: *variable*, *symbolic notation*, and *multiple representations*.

Variable. The notion of "variable" is of primary importance in algebra, as it forms the basis of generalizations. Research has identified reasons why students have difficulty understanding and using variables. When adult education instructors understand, look for, and recognize these problems, they can begin to help students overcome them.

Students have difficulty discriminating among the different ways letters may be used. First, in arithmetic letters are first encountered in formulas (for example,), which are generally provided as procedural guides. The student is asked to substitute the appropriate quantities to determine the perimeter, area, or volume. Letters in the formulas are actually variables, but they are rarely discussed as such. Second, a letter can represent a specific number that is currently unknown but should be determined or "found" (e.g.,). Third, a letter can represent a general number (such as), which is not one particular value. Finally, letters can represent variables (such as in), each of which represents a range of unspecified values along with a systematic relationship among them (Kieran 1992, p. 396). Teachers may assume that students can easily navigate among these different uses and meaning of letters, but interviews with adult students indicate this is not necessarily so (Jackson and Ginsburg 2008). Understanding the use of variables is crucial for students' success in algebra.

Some students have been found to believe that letters represent particular objects or abbreviated words because of their alphabetic connection (e.g., that represents "three dogs") or that a letter is a general referent (John's height is 10 inches more than Steve's height:) (MacGregor and Stacey 1997). Again, teachers need to pay close attention to such errors.

Symbolic notation. Some students have difficulty understanding that some of the symbolic notation familiar from arithmetic has different meanings and uses in algebra. Again, while the research studies were conducted with children and teens, the patterns of performance are similar for adult learners, and the findings can inform adult algebra instruction.

In arithmetic, the equal sign is commonly seen when the task requires a numeric operation, as in $4 + 8 =$ __. Many students perceive that the equal sign should be read, "And the answer is …." In algebra, the symbol = represents *equivalence* between two expressions (as in); *equality* (as in); or a *functional relationship* (as in). When the equal sign is perceived as signifying a symmetric balance between two quantities (on either side of the sign), then the need to perform the same operation on both sides of the equation becomes meaningful. Students' success in solving equivalent equation problems has been found to be related to the sophistication of their understanding of the relational aspect of the equal sign (Alibali et al. 2007; Knuth et al. 2006; Rojano 2002).

Other common mathematical symbols differ in use and meaning in arithmetic and algebraic contexts, and some students have difficulty discriminating between them. For example, the plus (+) and minus (–) symbols signify executable operations (addition, subtraction) in arithmetic, but they also indicate negative and positive numbers as well as operations in algebra (Gallardo 2002; Keiran 1992; Vlassis 2004; as cited in Kieran 2007).

Multiple representations. Multiple representations are used to describe, understand, and communicate generalizations algebraically. These include symbols, tables, graphs, and verbal descriptions. The perceptual aspects of different representations have an impact on students' reasoning. There are developmental trajectories for the different representations, with intuitive knowledge and previous experiences contributing to understanding (Brenner et al. 1997; Friel, Curcio and Bright 2001; Nathan and Kim 2007; Swafford and Langrall 2000). Each representation has its benefits and limitations, requiring a consideration of their tradeoffs when deciding their appropriateness for the task at hand. Often, representations are addressed separately during instruction, with an extended unit focused on symbol manipulation, a separate unit on patterns and tables, and a further unit on graphing. Students perceive that the different representations reflect separate and unrelated procedures and content. One of the goals of algebra instruction is to provide students with experiences that will enable them to make connections among the different representations and build flexibility in moving across them.

Introducing Elements of Algebra Instruction Early

In response to students' difficulties transitioning from arithmetic to algebra when the two subjects are disconnected by time and content, K–12 educators have begun integrating algebraic ideas and representations into all levels of arithmetic learning, thus encouraging the gradual development of algebraic reasoning. Researchers have explored the development of algebraic reasoning in younger students still learning arithmetic and have found that they can reason algebraically and that algebraic work may even facilitate their understanding of arithmetic (Carpenter and Franke 2001; Schifter 1999). Reform mathematics curricula, developed over the last 20 years and implemented widely over the last 10 years, all integrate algebra throughout the grades, following the lead of the National Council of Teachers of Mathematics standards (1989, 2000), among others. These curricula generally include a functions approach, relying heavily on modeling real work events or activities.

The What Works Clearinghouse (WWC) used a standard of rigorous methodology to examine numerous reports and studies of curriculum interventions to see if claims of effectiveness were supported. Of particular interest is the performance of groups of students using one of the reform curricula compared with that of control groups using traditional curricula.

For elementary-school math, WWC (2007a) examined 237 intervention studies and found 9 studies of 5 curricula that met its methodological criteria. One curriculum, Everyday Mathematics, was found to have potentially positive effects on mathematics achievement, with no overriding contrary evidence. Among the features of this program is a continuous strand of algebra from kindergarten through sixth grade, intertwined with other content. For example, in fourth through sixth grades, the topics include patterns, sequences, and functions; functions and multiple representations; solving number sentences by algebraic manipulations; grouping symbols and order of operations; simplifying expressions; and proportion (The University of Chicago School Mathematics Project 2004).

At the middle-school level, defined as grades 6 through 9, the WWC (2007b) found 21 studies of 7 curricula that met its standards. Five curricula showed positive or potentially positive effects. The WWC notes that one of these programs, the University of Chicago School Mathematics Project (UCSMP Algebra), "highlights applications, uses statistics and geometry to develop the algebra of linear equations and inequalities.… [and] emphasizes graphing, while manipulation with rational algebraic expressions is delayed until later courses" (WWC 2007b, p. 1).

These findings indicate that integrating elements of algebra while children are learning arithmetic is effective in promoting positive learning outcomes. Algebra curricula that focus primarily not on symbol manipulation, but rather on multiple representations and a functional approach have also proven effective. Although these studies were not conducted with adults, they provide a basis for reconceptualizing how algebra might be taught to enhance the development of adults' algebraic reasoning.

This approach seems particularly salient as a way of ameliorating adults' difficulties in making the transition from arithmetic to algebra. Adults returning to school arrive with patchy knowledge, varied experience studying mathematics and algebra, and limited time to participate in education. Putting off encountering algebra until they master all arithmetic content often means that adults are effectively excluded from studying algebra.

Mathematics Instruction in Adult Education Today

Presently, the goal of most mathematics instruction in a typical adult education program is students' acquisition of the ability to complete arithmetic procedures with whole numbers, fractions, decimals, and percents on paper without error (Schmitt 2000). It seems to be generally accepted that a certain level of skill with these procedures must be attained before any real applications or algebraic concepts can be understood. This sequence is reinforced by most materials published for use in adult education and by the most common tests used to assess student progress.

Studying algebra is usually delayed until the final stages of preparation for the GED math test. Algebra is assumed to be difficult and taught as if it were completely irrelevant to real life or to any prior mathematics learning. Thus, algebra is generally approached with a great deal of anxiety by students and teachers alike. GED-prep algebra instruction tends to focus on elementary topics of symbol manipulation, simplifying expressions, and solving equations. This "quick fix" approach relies on memorizing sequences of steps and does not present a coherent picture. Students seldom gain the conceptual understanding and reasoning ability needed for the successful pursuit of further goals. Not surprisingly, this minimal instruction also is not sufficient to permit students to opt out of developmental courses in community colleges when they take placement tests (e.g., ACCUPLACER, COMPASS).

Adult high school, adult diploma, and external diploma programs offer full-credit high school algebra courses. Students receive credit for the courses and must also pass the state high school exit exams to earn a diploma. Many teachers for these courses are not credentialed to teach mathematics and are not comfortable teaching algebra to high-risk students. Their instruction follows their own experience with algebra, in which they focused on symbols (x's and y's) and the seemingly incoherent rules that guide procedures using them in the abstract domain. It is a theoretical approach in which word problems are injected into some procedural lessons to offer practice in the new skill, serving mainly as puzzle-like extensions, not serious problem solving. As in the traditional high school curriculum, course objectives are aligned with prerequisites for the next mathematics course and ultimately the study of calculus, whether or not the students are likely to pursue calculus.

The nature of typical algebra instruction is not surprising, given that the rote elements of algebra described above are also the main components of assessments presently used in determining the proficiency level of candidates for educational advancement (state high school exit examinations and college placement tests). The multiple-choice format common in these tests makes narrow skills easier to assess and score than the understanding of broad concepts. The impact on instruction is predictable — to help students advance, teachers teach what they think will be tested, and the curricula narrow to meet the demands of tests.

The development of automaticity in manipulating abstract symbols is an essential part of academic preparation for students intending to major in science and mathematics, We question, however, whether this is an appropriate emphasis for students who have different goals or endpoints for their education. Everyday competence and upward mobility in the workplace require some symbol manipulation skills, but also require deeper understanding, reasoning ability, and connections to the demands of the real world.

Changes in Practice

There are indications that the general practice as described above is changing. For example, some states have written their adult education mathematics standards to reflect the requirement for understanding concepts as well as completing procedures (Arizona, Massachusetts, New York, Florida, Wisconsin and others that have adapted the Equipped for the Future [EFF] standards to their needs). Curricula introducing algebraic elements and concepts early in students' study of math have been developed and increasingly are being used nationally (EMPower, GED Math Problem Solver, EFF Curriculum Framework). With respect to assessment of learner gains leading up to the GED, the Massachusetts Adult Proficiency Test is built on a framework that embeds the strand of algebraic thinking at all levels. The CASAS test also is moving in that direction with the publication of math standards that will soon be reflected in its assessments.

The GED and Transition to College

In adult education, passing the GED test or earning a traditional high school diploma has been an important end goal for most students. A recent report estimates that only about 27 percent of those with a GED enter college, while 63 percent of those with a traditional high school diploma do (Reder 2007, p. 11). In an era, however, when some postsecondary training is becoming critical for more occupations, we can anticipate that a greater percentage of students will have goals that extend further and that more adult education students will choose to enroll in college.

Data currently show that a slightly larger percentage of GED graduates (25 percent) than traditional high school graduates (19 percent) are enrolled in developmental courses in community college and that remedial mathematics is the course most often taken by both groups. In fact, the percentage taking remedial math is twice the percentage taking remedial reading or writing classes (Reder 2007, p. 22). Several programs have been created to address this issue.

The New England ABE-to-College Transition Project provides assistance to those with high school diplomas or GEDs who need refresher courses and/or strategies for passing college placement exams and for eventual success at the college level. Another example is the adult secondary education algebra sequence in the Portland Adult School (Maine), where students have demonstrated exemplary rates of success when taking college mathematics placement tests. Remarkably, it is also a program where attention is paid to student attitudes and interests, time is taken to address conceptual underpinnings, and connections are made to real applications of algebra.

Revising Mathematics Instruction in Adult Education

When one views algebraic thinking as a sense-making tool, connections tend to become more horizontal than vertical. That is, rather than just building toward the next mathematics course, algebra becomes an empowering tool that facilitates deeper understanding of operations with numbers in arithmetic and enables interpretation of the numbers and graphs that adults encounter in daily life. From this perspective, the "basics" must evolve and new priorities emerge; some topics gain importance and others fade.

We suggest that adult education programs make two adjustments that have been effective in K–12 education and are being introduced in many community college courses: (1) integrate elements of algebraic thinking into arithmetic instruction and (2) reorganize formal algebra instruction to emphasize its applications.

Integrate Elements of Algebraic Thinking into Arithmetic Instruction

NCTM (2000, 2006), the Adult Numeracy Network (2005), and the Equipped for the Future Performance Continuum (National Institute for Literacy 1996) recommend that all content areas of mathematics (including algebra) be included at all levels of mathematics instruction and that conceptual understanding and problem solving be integrated into efforts to promote fluency with procedures. The introduction of early algebraic thinking into arithmetic instruction eases students' transition to a formal algebra course while also helping them make sense of arithmetic procedures and applications.

As part of their "algebrafication strategy," Blanton and Kaput (2003) recommend that elementary school teachers learn to identify and create opportunities for algebraic thinking as part of regular instruction. Kaput's three strands of algebra — generalizing using symbols, expressions, and equations; functions and relationships between quantities using the representations of symbols, equations, tables and graphs; and modeling real-world phenomena — all have seeds in arithmetic study. Explicitly calling attention to these strands of algebra is the hallmark of many innovative, effective curricula in K–6. In addition, the curricula require students at all levels to think flexibly and grapple with problems more complex than mere exercises. Mimicry ("do it this way") is replaced with exploration of multiple approaches ("you can do it more than one way") to solving problems.

For example, acquiring symbolic language by using letters in place of numbers is one skill that can be introduced early in arithmetic. Letters can mean "what number?" when basic facts and inverse operations are learned (e.g., $5 + x = 13$). They can also serve as placeholders for "any number" when making generalizations that are true for whatever number is inserted. For example, the following principles help students see regularities in arithmetic computations: $n + 0 = n$, or, where. Letters also indicate which values can change or vary in relationships such as geometric formulas. For example, in the formula for the perimeter of a square, $P = 4s$, the rule does not change, but the values can; the perimeter is 4 times whatever s is.

An important process used in algebraic thinking, recognizing patterns and making generalizations, can be introduced and practiced as early as when reviewing multiplication facts with adults. The array of facts below enables students to see the patterns that occur and the principles (structure) that can be derived from them in the multiplicative domain.

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

That the number 1 is the multiplicative identity element () is clear because the rows and columns following the 1's are exact images of the headings. The rows and columns of zeroes make it clear that multiplying by zero results in zero. (). The left-to-right diagonal of perfect squares (1, 4, 9, etc. that are the result of multiplying a number by itself) separates the array into two identical halves that are mirror images of each other. Noticing, for example, the two sets of 12's that would fall on each other if the paper were folded on that diagonal demonstrates that $3 \times 4 = 4 \times 3$ and $2 \times 6 = 6 \times 2$, examples of the commutative property of multiplication ($a \times b = b \times a$). The fact that multiplication and division are inverse operations can be illustrated when using the table to divide. (For example, to divide 56 by 8, start at the 56 in the row headed by the 8 and move up to find the 7 in the column heading.)

Properties of numbers and operations are critical in forming the basis for the algebraic methods used in simplifying expressions and solving equations. The identity elements for multiplication (1) and addition (0) along with the "undoing" effect of inverse operations are cornerstones of the logic for solving algebraic equations. (Striving for 1's and 0's is key to isolating the variable on its side of the equation.) Moreover, the cohesive structure that results from this kind of thinking helps students make sense of the various procedures as they learn them. Students learn that getting the right answer, while important, is not the only goal of their study. Reasoning "why" a procedure works as well as knowing "how" to do it is essential when preparing for algebra.

Using mental math and estimating when computing are important life skills for all adults. Developing these skills also relies heavily on the ability to recognize the structure of numbers and operations. For example, the variety of possible methods of computing 98 x 12 allows students to reason about place value and the operation of multiplication.

1. Estimate as $100 \times 10 \approx 1000$
2. Exact answer by using distributive property in $12(90 + 8) = 1080 + 96 = 1176$ (Good only for those who know their 12's)
3. Use polynomial multiplication $(90 + 8)(10 + 2) = 900 + 80 + 180 + 16 = 1176$ (Addition too complicated for mental math)
4. OR use $(100 – 2)(10 + 2) = 1000 + 200 – 20 – 4 = 1176$ (Insight leads to an easier computation)

Note the similarities in #2 and #3 with the traditional algorithm. Reflect on why this works (make the algebra explicit and stress that these are valid methods, not just tricks). Different situations lend themselves to different strategies.

When students discuss whether certain estimation and mental math techniques will always work, they strengthen their reasoning skills and prepare for the algebraic processes of generalization, pattern recognition and symbol manipulation.

Modeling real-world situations with mathematical structures is another way to introduce algebra during arithmetic instruction. Teachers can extend arithmetic word problems by varying the parameters, so that a problem with a single answer becomes an algebraic expression or equation. For example, consider the following problem:

> If William can save $15 a week toward the purchase of a $239 iPod, how many weeks will it take him to save enough?

By varying the amount he can save each week, this single problem can become a sequence of problems: "What if he could save $20 a week? Or $25?" After completing several problems, students are encouraged to reflect and generalize about what they have done. In effect, these questions move students from thinking in purely concrete terms to a more abstract mode involving a variable. Teachers can guide this process of moving from natural language to symbolic language, making a table, drawing a graph, and/or writing an equation that captures the relationship.

These classroom practices are examples of how teachers can weave an emphasis on conceptual understanding and reasoning into the development of basic skills in a way that fosters algebraic reasoning. This gradual but systematic introduction to the elements of early algebra elevates the study of arithmetic to more than just a warmed-over version of prior school experience and can reduce the intimidation students may feel when eventually taking a formal algebra class.

Reorganize formal algebra instruction to emphasize its applications. A *modeling* or *functions* approach in a formal algebra course can be helpful, especially for adults. Rather than building from abstract symbols and their manipulation to more abstractions, the modeling approach builds from contextual relationships between quantities that vary and are expressed as symbols (Chazan 2000). Real situations are used to motivate students and strengthen their understanding of much of what is included in a traditional algebra course. Situations representing how algebra is used in the real world can appeal to adult learners, who have more immediate career goals and appreciate the opportunity to apply their mature "common sense" to the learning process.

In a data-driven modeling approach, problem situations are defined by a set of data (input, output). For example:

> In an electronics store, a shipment of 400 music players is received, the stock is displayed on the floor, and sales ensue. The number remaining in the inventory is recorded weekly in a table, so that the manager can decide when to order the next shipment to ensure a constant supply.

The data points (week number, quantity in inventory) give students the information to analyze. They can plot these points on a graph (or enter them into a spreadsheet or a graphing calculator) and make judgments based on the general shape of the data. Assuming these particular data form a linear pattern, the students construct a "line of best fit" and notice particular aspects, such as the rate at which the inventory changes each week, which can be described abstractly as the slope of a line. They then use variables to write a mathematical equation that describes the relationship succinctly. In this way, students connect data to a verbal description of its quantitative relationship and its symbolic and graphical representations. Finally, they can use both the graph and the equation to predict values for data points in the future and decide when to order the next shipment.

Even though problems like this inventory example cannot claim to be "real life," in that they are packaged neatly for classroom use, they represent a huge improvement over the hopelessly contrived word problems in a traditional algebra course. When appropriate technology is available to assist, problems can reflect the ways algebra is used in real situations, such as business (e.g., predicting a break-even point after initial investment), environmental science (determining future needs for waste disposal sites), and medicine (recommending dosages and times in light of absorption time).

Using the modeling process for linear relationships, as well as repeating it for more complex functions (quadratic, exponential), requires that students learn and practice the fundamental rules of manipulating abstract symbols, which are based on the properties of numbers and operations first encountered in arithmetic. In this way, the bulk of the content of traditional algebra courses is studied for a purpose within contexts that have immediacy for adult students.

The modeling approach does not eliminate practice with symbolic manipulation, but offers a reason for it. By using authentic applications in an algebra course while also paying attention to theoretical and conceptual foundations, educators can achieve a balance and help to blur the distinction in preparation for academic versus vocational pursuits (Bailey 1998). The learning needs of both students pursuing science and math goals and those with other end-points in mind can be accommodated by this approach.

The Role of Assessment

Pedagogical principles for adult learners and several research studies suggest that the changes proposed in this section can increase adult students' access to and success in algebra. How effective, though, will this approach be in preparing students for the standardized tests they must take to advance? Most current assessments are structured to favor the traditional approach to both arithmetic and algebra (using multiple-choice questions narrowly focused on computation and symbol manipulation) and thus could be a potential barrier to widespread adoption of these changes. In fact, a Rand Education study (Le et al. 2006) showed that high school teachers may be hesitant to commit to reform methods because they believed that such changes were less likely to lead to high scores on accountability measures. In other words, teachers may continue to believe that "teaching to the test" is the best way to prepare their students.

Fortunately for adult education instructors, the GED mathematics test does reflect the instructional emphases recommended in this paper. The cognitive specifications for the 2012 version of the test require that only 20 percent of the items will be procedural in nature, while 30 percent will be conceptually based and 50 percent will reflect applications, modeling, and problem solving. Examinees will be allowed to use a calculator for the entire test. In terms of content, 30 percent of the items will come from the area of "Algebra, Functions, and Patterns," where benchmarks focus on concepts emphasized in the modeling approach to algebra. Since the GED test is the high-stakes assessment of interest to most adult students, the strong alignment between what is assessed and what is taught can generate student interest while promoting good instructional practice.

The alignment between assessment and high-quality instruction is not as strong in other mathematics tests that adult students may face. These include assessments that classify and evaluate student progress (Test for Adult Basic Education, or TABE, and (Comprehensive Adult Student Assessment Systems or CASAS) and those designed to place students in the most appropriate college mathematics class (such as ACCUPLACER and COMPASS). In these tests, the procedural aspects of mathematics seem to take precedence over the conceptual understanding that is fostered by the recommendations in this paper. This reality creates a perceived dilemma for teachers, who recognize that adult students, whose immediate goal is to perform well on these tests, need more than procedures to achieve success in the long run.

Some research studies give reason for optimism about students' likely success with traditional assessments and course work, even when their instruction emphasized a broader range of knowledge than the assessments could detect. Boaler (1998) reports that, on a national standardized test, adolescents who learned mathematics through realistic projects and then spent a short time learning test-taking skills outperformed those who learned through traditional methods more obviously aligned with the assessment. Merely taking a practice test before taking an actual placement exam resulted in higher student scores in a study by Revak, Frickenstein, and Cribb (1997).

Stone et al. (2006) report that high school students in experimental classes in career and technical education programs where explicit attention was paid to the mathematical principles involved in occupational curricula performed significantly better on the ACCUPLACER test than those in the control group, who simply used mathematical tools and procedures within the hands-on context. At the college level, Norwood (1995) examined performance levels for two mathematics courses in a sequence. She found that students taught using a project-based modeling approach in the first course were more successful in the next course in the sequence (taught the traditional way) than were students who had been taught traditionally in the first course. Ellington (2005) found that fewer students dropped out of a college algebra course focused on modeling than dropped out of a traditional college algebra course.

Although these research studies were small and represent only modest evidence, they suggest that an approach based on concepts and modeling can meet both student needs — to pass the required tests and to acquire the conceptual understanding and reasoning skills critical for their future success.

Supporting Change in Classroom Practice

Strengthening mathematics instruction in adult education requires several enabling conditions to support and spread a new approach. These include:

Building Public Awareness

For most people, the meaning of the word "literacy" does not include mathematical ability. In Australia and the U.K., official language has been expanded to "literacy and numeracy" to convey the broader meaning. Recognizing the critical role that numeracy plays is important at the highest levels to direct attention to it at the program level. And a fuller understanding that the study of algebra has become essential for success in many career fields— but need not be the arcane exercise it was previously—should provide new incentive for change.

Preparing Well-Trained Teachers

To improve adult mathematics instruction as described in this paper, teachers need to know mathematics content and be comfortable guiding students in interactive study. Teacher knowledge of mathematics and the particular mathematical knowledge needed for teaching (including developmental trajectories and common misconceptions) have been shown to be factors in student learning as early as first grade (Hill, Rowan and Ball 2005). Passing an algebra course is not adequate preparation for teaching one. That said, only about 6 percent of those teaching mathematics to adults have the credentials to do so and thus most may well lack the confidence to use innovative curricula effectively (Gal and Schuh 1995; Ward, 2000).

Attempts to improve teacher preparation have begun. Along with their new standards, several states have sponsored staff development programs aimed at increasing teacher knowledge of both mathematics and mathematical pedagogy (Bingman and Schmitt 2008). A professional development pilot program that is part of the Adult Numeracy Instruction Project sponsored by the Department of Education's Office of Vocational and Adult Education also will address these issues. Technology also can be an asset in spreading staff development efforts.

Aligning Assessment and Instruction

When changes are made to instruction, a review of the related accountability and assessment structure must follow. As more states rewrite standards to broaden the mathematics curriculum by including algebraic thinking and conceptual understanding at all levels, a mismatch is occurring between these standards and national assessments. Assessments encouraging students to demonstrate more than computation skills would help motivate teachers to make the recommended changes in instruction.

Making Institutional Adjustments

Instruction that emphasizes the development of reasoning and meaning-making is facilitated by enabling students to examine and reexamine extended activities and investigations collaboratively. Students working individually are limited in their opportunities to move beyond the traditional emphases of algebra instruction. Therefore, programs should consider using managed enrollment to support the curricular suggestions for algebra discussed in this paper.

Research to Advance Understanding and Practice

Several significant aspects of adult math learning in general and algebra learning in particular could benefit from research. These include studies of the following.

Adult Math Learning

Studies of children and adolescents showing their difficulty transitioning from arithmetic to algebra and with certain algebraic concepts should be replicated with adult learners. Adults represent a significantly different population and already may have encountered algebra in school. Among the research questions: Do student proficiency outcomes vary according to different models of adult algebra instruction? How do adults perceive and respond to an instructional shift from an emphasis on procedures to an emphasis on sense making? Do they feel anxiety or relief with this shift? Do adults' previous, perhaps negative, experiences with algebra affect new algebra learning and their willingness to engage? How?

Persistence and Goal Achievement

Are there relationships between implementing algebra throughout the adult numeracy curriculum and students' persistence, self-image, perceptions of progress toward their goals, and success on the GED math test? How successful are learners who study algebra in adult education settings (including transition models) in placing out of or completing developmental algebra courses in community colleges? How successful are such learners in being accepted into and completing workforce training or community college certification programs?

English-Language Learners

How do issues of language and culture affect algebra learning among immigrants, including those with and without school experience in their native countries? Does a modeling approach to algebra instruction put more of a demand on English-language skills than instruction in procedural skills? Does this approach facilitate the development of mathematics and language skills in tandem?

Special Populations

What effect does an early emphasis on sense making rather than on procedures have on those with learning differences or disabilities? Does an approach that builds on multiple representations enable learners with specific limitations to use and master those representations accessible to them, so they can succeed in algebra after being unsuccessful previously?

Summary

We know that proficiency in science and mathematics is critical for our country's global competitiveness. Recent studies reveal the growing demand for educated workers and emphasize the urgency to make improvements in education that extend beyond the K–12 system to include lifelong education. Those lacking mathematical proficiency face limits to their participation in our society. By increasing access to and success in the study of algebra, adult education programs can make a significant step in meeting this challenge.

In adult education, algebraic thinking can be a sense-making tool that introduces coherence among mathematical concepts for those who previously have had trouble learning math. Further, a modeling approach to algebra connects mathematics and the real world, demonstrating the usefulness of math to those who have seen it as just an academic exercise.

This paper recommends two significant and necessary changes in our concept of algebra and the ways we approach it in education: a shift from thinking of algebra as one course to thinking of it as a content strand integrated into arithmetic instruction and a shift from thinking of algebra as merely manipulation skills to thinking of it as a means of representing and analyzing real situations. This fresh look at algebra and mathematics can result in instruction that appeals to mature students by addressing their needs in pursuit of a career and in their daily life by actively involving them in solving meaningful problems.

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